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Solving the Problem of Linear Programming with Cost Coefficients and Grey Resources Using the Primal Simplex Algorithm

Farid Pourofoghi*

Department of Mathematics, Payame Noor University, Tehran, Iran; f_pourofoghi@pnu.ac.ir

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Abstract

In most real-world issues, we are dealing with situations where accurate data and complete information are not available. One way to deal with these uncertainties in real life is to use Grey System Theory (GST). In this paper, a linear programing problem in a grey environment with interval Grey Numbers (GN) is considered. Most of the proposed methods for solving grey linear programing problems are done by using GN whitening and turning the problem into a common linear programing problem. However, in this paper we seek to solve the grey linear programing problem directly without turning it into a regular linear programing problem in order to maintain uncertainty in the original problem data in the final answer. For this purpose, by proving the desired theorems, we propose a method based on the initial simplex algorithm to solve grey linear programing problems. This method is simpler than the previous methods. We emphasize that the proposed concept is useful for real and practical situations. To illustrate the efficiency of the method, we solve an example of Grey Linear Programming (GLP).

Keywords: Uncertainty, Grey interval numbers, Grey linear programming.

1 | Introduction

One of the achievements of operations research is to help managers make the right decisions. In the meantime, one of the operations research techniques that helps managers to achieve their goal in the optimal form (maximum or minimum) of a linear function under constraints of linear equations or inequalities is the technique of Linear Programming (LP) in definite conditions [1], [2]. In a typical LP model, all input parameters of the model are known values (fixed and non-probable). In real cases, this assumption rarely applies. The LP model is usually formulated to select some future activities. As a result, the parameters used will be based on the prediction of future conditions, which will inevitably include degrees of uncertainty [3], [4]. Due to the complexities and scarcity of information in the real world, in recent decades, the design of methods for recognizing, modeling and managing uncertainty in the behavior of systems has become one of the fascinating scientific topics and various theories and methods such as fuzzy set theory and probability theory for The study of uncertain systems has

Corresponding Author: f_pourofoghi@pnu.ac.ir

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been developed [5]. In fuzzy set theory, experts are used and in probability theory, sampling is used. However, if the number of experts and the level of experience are so low that it is not possible to extract membership functions or the number of samples is low, these theories can no longer be used. To address such issues, Professor Joe Lang Deng first proposed a different model in 1982 called the Grey System Theory (GST) [3]. Grey Linear Programming (GLP) is a model of analysis of Grey Systems (GSs) for decision making in conditions of uncertainty, which is an extension of the conventional LP method [6]. The following can be considered of the methods proposed by researchers to solve the problem of LP with grey parameters. The first category is methods that use the concept of Grey Number (GN) whitening to solve LP problems with grey parameters. In this method, the grey information of the GLP problem is written as white or definite (real) numbers using the GN whitening methods and the GLP problem is written as the ordinary LP problem. In these methods, the answer to the initial problem is obtained by solving several ordinary LP problems. Among these methods, we can mention [6]-[12]. These methods can be used for any kind of problem and real number operators. However, some input information may be lost in the parameter whitening process and the resulting answer may not be the original answer. Also, in these methods, the final answer is not necessarily grey and the uncertainty in the input data is not well reflected in the output answers. The second category is the method of finding the answer to the GLP problem based on the use of overlapping numbers and inverse concepts of the grey matrix. This method is generally a good idea for solving GLP problems [13], [14]. Excessive calculations and failure to achieve the stop condition after finding the answer are the two main drawbacks of this method. The third category is a method based on the concepts of grey prediction [15]. In this method, we first obtain the desired values using the grey prediction method and then solve the problem of ordinary LP obtained. The fourth category is a method that uses the display of GNs in the form of intervals and the ranking of intervals to solve GLP problems. In this method, by converting the objective function of the GLP problem into several objective functions and defining the constraints related to each objective function, we separately turn the problem into several ordinary LP problems and solve each one [10], [16]. These methods are able to solve GLP in different conditions. Different sequence relationships are used to compare GNs to determine the input or output variable. Some researchers use the concepts of LP solution of interval numbers [6], [17]. The problem with these methods is that they ignore the grey space of the problem. In these methods, they solve the problem by using the upper and lower boundaries of the answer space, which in some cases may not have one of the problems. The fifth category is a method in which an attempt is made to solve the GLP problem directly and without the need to bleach the parameters of the problem using the simplex method, thus reflecting the uncertainty in the input data better in the final answer [18]. In this method, for comparing GNs, simultaneous comparison of center and degree of grey is used, which has a better performance in distinguishing between GNs. The disadvantage of this method is that it is used only to solve GLP problems with the grey objective function. In this paper, an algorithm such as the initial simplex algorithm is proposed to solve interval GLP problems, for which we do not need to convert the GLP problem to the classical LP problem.

The article is presented in 5 sections. After the introduction, in Section 2, the theory of the GS and the concepts related to interval GNs and their comparison are stated. In Section 3, the problem of GLP problems is introduced. In Section 4, the simplex method for GLP problems (with objective function coefficients and grey sources) and the proposed algorithm are presented. In Section 5, we present and solve a numerical sample to express the effectiveness and understanding of the proposed method. Finally, Section 6 contains the conclusion.

2 | GSs Theory

Various theories have been proposed to deal with the uncertainties in various subjects of real-life everyday life, including fuzzy system theory, probability theory, and so on. GSs theory is one of the methods proposed to study the uncertainty of problems that are highly inaccurate due to limited data and limited information. The GS is defined as a system that contains uncertain information [19]. GS theory has attracted many researchers [20]-[29].

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In this section, the necessary definitions for studying the GS are considered.

Definition 1. The interval GN is defined as follows [30].

$$\otimes x \in \left[\underline{x}, \overline{x}\right] = \left\{t \middle| \underline{x} \le t \le \overline{x}\right\},\ \underline{x} \le \overline{x}. \tag{1}$$

Definition 2. [31] Let $\otimes x_1 \in [\underline{x}_1, \overline{x}_1]$ and $\otimes x_2 \in [\underline{x}_2, \overline{x}_2]$ be two interval GNs.

I. For any interval GN, the center $\otimes \hat{x}$ is defined as;

$$\otimes \hat{x} = \frac{\underline{x} + \overline{x}}{2}. \tag{2}$$

II. For any interval GN, the length $\ell(\otimes x)$ is defined as;

$$\ell(\otimes x) = \left|\overline{x} - \underline{x}\right|. \tag{3}$$

III. For any interval GN, the degree of greyness $g^{\circ}(\otimes x)$ is defined as;

$$g^{\circ}(\otimes x) = \frac{\ell(\otimes x)}{\otimes \hat{x}},\ \otimes \hat{x} \ne 0. \tag{4}$$

IV. The main arithmetic operations can be defined on GNs.

$$\otimes x_1 + \otimes x_2 = \left[\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2\right]. \tag{5}$$

$$\otimes x_1 - \otimes x_2 = \otimes x_1 + \left(-\otimes x_2\right) = \left[\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2\right]. \tag{6}$$

$$\otimes x_1 \times \otimes x_2 = \left[\min\left\{\underline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2\right\}, \max\left\{\underline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2\right\}\right]. \tag{7}$$

$$\frac{\otimes x_1}{\otimes x_2} = \otimes x_1 \times \otimes x_2^{-1} = \left[\min\left\{\frac{\underline{x}_1}{\underline{x}_2}, \frac{\underline{x}_1}{\overline{x}_2}, \frac{\overline{x}_1}{\underline{x}_2}, \frac{\overline{x}_1}{\overline{x}_2}\right\}, \max\left\{\frac{\underline{x}_1}{\underline{x}_2}, \frac{\underline{x}_1}{\overline{x}_2}, \frac{\overline{x}_1}{\underline{x}_2}, \frac{\overline{x}_1}{\overline{x}_2}\right\}\right] \quad 0 \notin \left[\underline{x}_2, \overline{x}_2\right]. \tag{8}$$

2.1 | Comparison of GNs

Comparing GNs with each other is very important for making the right decision in a grey environment. Dervishi et al. [21] have compared GNs in more detail. In this article has been used the concept of kernel and degree of grey to compare interval GNs.

Definition 3. [32] For two interval GNs $\otimes x_1$ and $\otimes x_2$:

$$\otimes \hat{x}_1 < \otimes \hat{x}_2 \Rightarrow \otimes x_1 <_G \otimes x_2. \tag{9}$$

$$\otimes \hat{x}_1 = \otimes \hat{x}_2 \Rightarrow \begin{cases} g^{\circ}(\otimes x_1) = g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 =_G \otimes x_2 \\ g^{\circ}(\otimes x_1) < g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 >_G \otimes x_2 \\ g^{\circ}(\otimes x_1) > g^{\circ}(\otimes x_2) \Rightarrow \otimes x_1 <_G \otimes x_2 \end{cases}. \tag{10}$$

3 | GSs Theory

The problem with GLP is generally as follows.

$$\begin{aligned} &\text{Maximize } \otimes z =_G \sum_{j=1}^{n} \otimes c_j \otimes x_j \\ &\text{Subject to} \\ &\qquad \sum_{j=1}^{n} \otimes a_{ij} \otimes x_j \leq_G \otimes b_i, \quad i=1,2,\ldots,m, \\ &\qquad \otimes x_j \geq_G \otimes 0, \quad j=1,2,\ldots,n, \\ &\otimes c_j, \otimes a_{ij}, \otimes x_j, \otimes b_i \in R(\otimes), \; i=1,2,\ldots,m, \; j=1,2,\ldots,n. \end{aligned} \tag{11}$$

In this section, we define LP problems involving GNs as follows:

$$\begin{aligned} &\text{Maximize } \otimes z =_G \sum_{j=1}^{n} \otimes c_j \otimes x_j \\ &\text{Subject to} \\ &\qquad \sum_{j=1}^{n} a_{ij} \otimes x_j \leq \otimes b_i, \quad i=1,2,\ldots,m, \\ &\qquad \otimes x_j \geq 0, \quad j=1,2,\ldots,n. \end{aligned} \tag{12}$$

Its matrix form will be as follows:

$$\begin{aligned} &\text{Maximize } \otimes z =_G \otimes C \otimes X \\ &\text{Subject to} \\ &\qquad A \otimes X \leq_G \otimes b, \\ &\qquad \otimes X \geq_G 0. \end{aligned} \tag{13}$$

So that:

$$\begin{aligned} &\otimes X = [\otimes x_1, \otimes x_2, \ldots, \otimes x_n]^T, \\ &\otimes C = \begin{bmatrix} \otimes c_1 & \otimes c_2 & \ldots & \otimes c_n \end{bmatrix}^T, \\ &\otimes b = \begin{bmatrix} \otimes b_1 & \otimes b_2 & \ldots & \otimes b_m \end{bmatrix}^T, \\ &A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & \ldots\ldots\ldots\ldots & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \\ &\otimes c_j, \otimes x_j, \otimes b_i \in R(\otimes), \; a_{ij} \in R, i=1,2,\ldots,m, \; j=1,2,\ldots,n. \end{aligned} \tag{14}$$

$$\begin{aligned} &\otimes Z \in \left[\underline{Z}, \overline{Z}\right], \qquad \underline{Z}, \overline{Z} \in R, \\ &\otimes x_j \in \left[\underline{x}_j, \overline{x}_j\right], \quad \underline{x}_j, \overline{x}_j \in R, \quad j=1,2,\ldots,n, \\ &\otimes c_j \in \left[\underline{c}_j, \overline{c}_j\right], \quad \underline{c}_j, \overline{c}_j \in R, \quad j=1,2,\ldots,n, \\ &\otimes b_i \in \left[\underline{b}_i, \overline{b}_i\right], \quad \underline{b}_i, \overline{b}_i \in R, \quad i=1,2,\ldots,m, \\ &a_{ij} \in R, \quad i=1,2,\ldots,m, \; j=1,2,\ldots,n. \end{aligned} \tag{15}$$

Definition 4. Any vector $\otimes x$ of interval GNs that satisfy the constraints of the GLP in *Eq. (12)* is called a feasible solution. Assume that Q the set of all feasible solutions to the GLP in *Eq. (12)*. Then $\otimes x_0 \in Q$ is said to be an optimal solvable solution to the GLP, if for all $\otimes x \in Q$; $\otimes C \otimes x \leq_G \otimes C \otimes x_0$.

3.1 | Basic Feasible Solution

Consider the GS in *Eq. (13)*, where A is a $m \times n$ matrix, $\otimes b$ and $\otimes x$ are an m vector and n vector, respectively, satisfying $rank(A, \otimes b) = rank(A) = m$. After possibly rearranging the columns of A, let $A = [B, N]$, where B is a $m \times m$ invertible matrix and N is a $m \times (n-m)$ matrix. The solution $\otimes X^T = [\otimes x_B^T, \otimes x_N^T]$ to the equations $A \otimes X =_G \otimes b$, where $\otimes X_B = [\otimes x_{B_1}, \otimes x_{B_2}, \ldots, \otimes x_{B_m}]^T$ and $\otimes X_N =_G \otimes 0$, is called a basic solution of the system. If $\otimes x_B \geq_G \otimes 0$, then $\otimes X$ is called a basic feasible solution of the system and the corresponding grey objective value is $\otimes z =_G \otimes C_B \otimes X_B$, where $\otimes C_B = [\otimes c_{B_1}, \ldots, \otimes c_{B_m}]$. Now, corresponding to every index $1 \leq j \leq n$, define: $y_j = B^{-1} a_j$ and $\otimes z_j =_G \otimes c_B\, y_j$. Observe that for any basic index $j = B_i, 1 \leq j \leq m$, we have: $\otimes z_j - \otimes c_j =_G \otimes c_B B^{-1} a_j - \otimes c_j =_G \otimes c_B e_i - \otimes c_j =_G \otimes c_j - \otimes c_j =_G \otimes o$.

Where e_i is the ith unit vector. Note that B is called the basic matrix and N is called the non-basic matrix. The components of $\otimes X_B$ are called basic variables and the components of $\otimes X_N$ are called non-basic variables.

Theorem 1. [24] If there is a basic feasible solution with grey objective value $\otimes Z$ such that $\otimes y_{0j} =_G \otimes Z_j - \otimes C_j <_G \otimes 0$ or $\otimes Z_j <_G \otimes C_j$ for some non-basic variable $\otimes x_j$, and $y_j = B^{-1} a_j \nleq 0$ $1 \leq j \leq n$, then it is possible to obtain a new basic feasible solution with a new grey objective value $\otimes Z'$ that satisfies $\otimes Z \leq_G \otimes Z'$.

Theorem 2. [24] If there is a basic feasible solution satisfying $\otimes y_{0k} =_G \otimes C_B B^{-1} a_k - \otimes C_k =_G \otimes Z_k - \otimes C_k <_G \otimes 0$, for some non-basic variable $\otimes x_k$, and $y_{ik} \leq_G 0$, $i = 1, 2, \ldots, m$, then the problem in *Eq. (12)* has an unbounded optimal solution.

Theorem 3. The basic feasible solution $\otimes X_B =_G B^{-1} \otimes b$, $\otimes X_N =_G \otimes 0$ is an optimal solution to the *Eq. (12)*, whenever $\otimes c_B B^{-1} a_j \geq_G \otimes c_j$, for all $j = 1, 2, \ldots, n$.

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Proof. Suppose $\otimes X_*^T = [\otimes x_B^T, \otimes x_N^T]$ a basic feasible solution is for the GLP *Eq. (12)*. So that $\otimes X_B =_G B^{-1} \otimes b$ and $\otimes X_N =_G \otimes \circ$, therefore, the optimal value of the objective function of *Eq. (12)* for the solution $\otimes X_*^T$ will be as follows:

$$\otimes z_* =_G \otimes C_B \otimes X_*^T =_G \otimes C_B \otimes X_B =_G \otimes C_B\ B^{-1} \otimes b.$$

On the other hand, for each basic feasible solution $\otimes x$, we have the following relation:

$$A \otimes X \leq_G \otimes b =_G A \otimes X =_G B \otimes X_B + N \otimes X_N.$$

Therefor;

$$\otimes z =_G \otimes C \otimes X =_G \otimes C_B \otimes X_B + \otimes C_N \otimes X_N =_G \otimes C_B\ B^{-1} \otimes b - \sum_{j \neq B_i} \left(\otimes C_B\ B^{-1}\ a_j - \otimes C_j \right) \otimes X_j.$$

And,

$$\otimes z =_G \otimes z_* - \sum_{j \neq B_i} \left(\otimes z_j - \otimes C_j \right) \otimes X_j.$$

According to this relation and Theorem 1, the proof is complete.

4 | Grey Linear Programing Problem Based on the Primal Simplex

Consider the GLP problem as in *Eq. (12)*. Suppose the matrix B is base, in this case we rewrite the GLP problem as:

$$\begin{aligned} &\text{Maximize} \otimes z =_G \otimes C_B \otimes X_B + \otimes C_N \otimes X_N \\ &\text{Subject to} \\ &\qquad B \otimes X_B + N \otimes X_N =_G \otimes b, \\ &\qquad \otimes X_B \geq_G \circ, \otimes X_N \geq_G \circ. \end{aligned}$$

Hence we will have:

$$\otimes X_B + B^{-1}\ N \otimes X_N =_G B^{-1} \otimes b \Rightarrow \otimes X_B =_G B^{-1} \otimes b - B^{-1}\ N \otimes X_N.$$

Therefor:

$$\begin{aligned} &\otimes Z =_G \otimes C_B \left(B^{-1} \otimes b - B^{-1}\ N \otimes X_N \right) + \otimes C_N \otimes X_N \\ &=_G \otimes C_B B^{-1} \otimes b - \otimes C_B B^{-1}\ N \otimes X_N + \otimes C_N \otimes X_N \\ &\otimes Z + \left(\otimes C_B B^{-1}\ N - \otimes C_N \right) \otimes X_N =_G \otimes C_B B^{-1} \otimes b. \end{aligned}$$

Table 1. Simplex table related to GLP problem

Basic Variables	$\otimes X_B$	$\otimes X_N$	Sources
$\otimes Z$	$\circ$	$\otimes C_B B^{-1}\ N - \otimes C_N$	$\otimes y_{\circ\circ} =_G \otimes C_B B^{-1} \otimes b$
$\otimes X_B$	I	$B^{-1}\ N$	$\otimes y_{\circ\circ} =_G B^{-1} \otimes b$

4.1 | The Primal Simplex Algorithm of GLP

Suppose that a basic feasible solution is also accompanied with a basis B and corresponding simplex table.

I. The basic feasible solution is given by $\otimes x_B =_G B^{-1} \otimes b =_G \otimes y_0$ and $\otimes x_N =_G \otimes 0$.

Then the grey objective function value will be $\otimes Z =_G \otimes c_B \cdot B^{-1} \cdot \otimes b =_G \otimes y_{00}$.

II. Calculate $\otimes y_{\circ j} =_G \otimes Z_j - \otimes C_j$, $j = 1,2,...,n$ $j \neq B_i$, $i = 1,2,...,m$.

Let $\otimes y_{0k} =_G \min_{1 \le j \le n} \{\otimes y_{0j}\}$.

III. If $\otimes y_{0k} \geq_G \otimes 0$, then stop; the solution is optimal.

IV. If $\otimes y_{0k} <_G \otimes 0$ and $y_{ik} \le 0$ $i = 1,2,...,m$, the problem has an unbounded solution.

V. If $\otimes y_{0k} <_G \otimes 0$ and there is $i = 1,2,...,m$ so that $y_{ik} > 0$, then determine an index r corresponding to a variable x_{Br} that leaves the basis as follows:

$$\frac{\otimes y_{r0}}{y_{rk}} =_G \min_{1 \le i \le m} \left\{ \frac{\otimes y_{i0}}{y_{ik}} \middle| y_{ik} > 0 \right\}.$$

VI. Pivot on y_{rk} and update the simplex tableau. Go to Step 2.

In the following, we will look at an example to demonstrate how the above Algorithm plays out in real life.

5 | Numerical Example

In this section, by presenting example method's efficiency to solve, is evaluated.

Example. Consider the GLP problem below.

$$\text{Maximize} \otimes z =_G \otimes[1,3] \otimes x_1 + \otimes[2,5] \otimes x_2$$

Subject to

$$2 \otimes x_1 + 3 \otimes x_2 \le_G \otimes[5,7],$$
$$3 \otimes x_1 + \otimes x_2 \le_G \otimes[3,6],$$
$$\otimes x_1, \otimes x_2 \ge_G \otimes 0.$$

Solve. The above problem is called the initial GLP problem. The standard form of the above GLP problem is considered as follows.

$$\text{Maximize} \otimes z =_G \otimes[1,3] \otimes x_1 + \otimes[2,5] \otimes x_2$$

Subject to

$$2 \otimes x_1 + 3 \otimes x_2 + \otimes s_1 =_G \otimes[5,7],$$
$$3 \otimes x_1 + \otimes x_2 + \otimes s_2 =_G \otimes[3,6],$$
$$\otimes x_1 , \otimes x_2 \geq_G \otimes 0.$$

Table 2. The first simplex table of the GLP problem.

Basic Variables	$\otimes x_1$	$\otimes x_2$	$\otimes s_1$	$\otimes s_2$	Sources
$\otimes z_0$	$-\otimes[1,3]$	$-\otimes[2,5]$	$\otimes[0,0]$	$\otimes[0,0]$	$\otimes[0,0]$
$\otimes s_1$	2	3	1	0	$\otimes[5,7]$
$\otimes s_2$	3	1	0	1	$\otimes[3,6]$

Table 3. The optimal simplex table of the GLP problem

Basic Variables	$\otimes x_1$	$\otimes x_2$	$\otimes s_1$	$\otimes s_2$	Sources
$\otimes z_0$	$\otimes[-\frac{5}{3},3]$	$\otimes[0,0]$	$\otimes[\frac{2}{3},\frac{5}{3}]$	$\otimes[0,0]$	$\otimes[\frac{10}{3},\frac{35}{3}]$
$\otimes x_2$	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\otimes[\frac{5}{3},\frac{7}{3}]$
$\otimes s_2$	$\frac{7}{3}$	0	$\frac{1}{3}$	1	$\otimes[\frac{2}{3},\frac{13}{3}]$

6 | Conclusion

In most real-world issues, we are dealing with situations where accurate data and complete information are not available. Therefore, LP is not accurate due to the assumption that the parameters are constant. Given that the theory of the GS is one of the approaches to deal with uncertainty. There is an urgent need to provide a suitable solution for GLP models. To date, many studies have focused on this topic, most of which are discussed in this article. Some of the proposed methods were not able to solve the GLP models in different conditions, while others did not provide the appropriate optimal solution. In general, the GLP problems transformed into one or a series of the classical LP problems and then obtained an optimal solution. As shown in the paper, an algorithm like well-known primal simplex algorithm for solving GLP problems is proposed, where it can solve the problem without converting to the classical LP problems. In future research, researchers could suggest a more accurate way to solve the GLP problem.

Conflicts of Interest

All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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